Interpreting and Projecting Mortality Trends for European Countries by Using the LD Model*

Futoshi ISHII†- Giampaolo LANZIERI‡

Introduction

Population projections are one of the demographic outcomes with higher relevance for various policies, especially in those countries where the pace of ageing is rapid. This applies particularly to the Member States of the European Union, where population projections are officially part of a regular process of assessment of the long-term sustainability of public finances (European Commission and Economic Policy Committee 2012). Assumptions on future mortality play there a key role for pension, health care and long-term care policies. In that exercise, the current methodology for mortality projections is based on the partial long-term convergence to an ultimate age mortality pattern, identified by means of an extrapolation from the latest trends of selected countries, forerunners in terms of mortality improvements, using the Booth-Maindonald-Smith variant (Booth et al. 2002) of the Lee-Carter model (Lanzieri 2009).

In general, improvement for adult mortality is understood as declining of the mortality curve. However, Ishii (2008) shows how adult mortality improvement in Japan could be better modelled by the shift-type than by the decline-type model, such as the Lee-Carter (LC) model (Lee and Carter 1992). The same scholar has proposed a new shift-type model for adult mortality, called the Linear Difference (LD) model, and applied it to the official population projection for Japan (National Institute of Population and Social Security Research 2012).

In this study, we apply the LD model to the adult mortality for several European countries to analyze the trends of mortality improvements. Through the comparison of the parameters

---

* This paper is released to inform interested parties about research work. The views expressed are exclusively those of the authors and not necessarily represent the views of neither the European Commission / Eurostat nor National Institute of Population and Social Security Research.
† National Institute of Population and Social Security Research, Tokyo (ishii-futoshi@ipss.go.jp)
‡ Statistical Office of the European Union (EUROSTAT), Luxembourg (giampaolo.lanzieri@ec.europa.eu)
for the LD model between countries, we try to elucidate the peculiar features of mortality in Europe, which are also applicable for projections purposes. These characteristics will be also compared to those of Japan, a benchmark country in the field of mortality. Given the variety of mortality patterns existing in Europe, this study is also an important test about the performance of the LD model.

1 Data and Methods

1.1 Data

For the purpose of the mortality study in this paper, we used mortality data for female from 1970 to 2010 by the Human Mortality Database (HMD) for European countries and those by the Japanese Mortality Database (JMD) for Japan. Mortality data for Croatia, Cyprus, Greece, Malta and Romania are not included in the HMD, so these countries are excluded in this study. We also excluded Slovenia neither, since the data are available only from 1983.

We mainly worked on $m_{x,t}$ functions where $x$ is age and $t$ is a calendar year. We extrapolated the mortality rates above age 110 fitting the two parameter logistic model

$$m_{x,t} = \frac{\alpha_t \exp(\beta_t x)}{1 + \alpha_t \exp(\beta_t x)}$$

that are the same method as used in the HMD (and JMD).

1.2 Formulation of Declining and Shifting Models

Regarding the mortality improvement as *declining* means that the value of log mortality declines over time for a fixed age. In this case, the mortality improvement rates for a age decreases, or equivalently, the differential of the mortality improvement rates by time is negative. On the other hand, the delay of the age that attains to a fixed value of the log mortality is considered as *shifting* of mortality curve. In this case, the age that attains to a fixed value of the log mortality increases, or equivalently, the differential of the age by time is positive.

Actually, the function that express the age that attains to a fixed value of the log mortality is the *inverse* function of the log mortality rates. Therefore, considering a *shift*-type model for the log mortality rates is identical to considering the *decline*-type model for the inverse function of the log mortality rates (Figure 1, 2).

Based on these observations, we can state the mathematical formulations putting special emphasis on the log mortality and its inverse functions, and the differential of them by time.

Let $X = [0, +\infty)$ be the space of age and $T = (-\infty, +\infty)$ be the space of time. In the following discussion for modeling mortality, we will work on $\mu_{x,t}$, the hazard function for exact age $x \in X$ at time $t \in T$. 

2
The log hazard function of mortality is expressed by $y = \lambda_{x,t} = \log \mu_{x,t}$, where $y \in Y = (-\infty, +\infty)$ is the value of the function. Then, the set $S = \{(x,t,y) | y = \lambda_{x,t}\}$ determines a surface in $\mathbb{R}^3$, called the log mortality surface. This is a conventional representation of the log mortality surface. In this representation, $y = \lambda_{x,t}$ would be considered as the height from the $X$-$T$ plane in $\mathbb{R}^3$.

Here, we consider another representation of the log mortality surface under a set of assumptions.

We assume that $\lambda_{x,t}$ is a smooth continuous function with respect to $x$ and $t$ defined on $X_0 \times T_0 = [0, \omega] \times [t_0, t_1] \subset X \times T$, where $\omega < +\infty$ is a finite maximum age for mortality models.

For the purpose of modeling adult mortality, we can further assume that $\lambda_{x,t}$ exhibits a strictly monotonic increase with respect to $x$ for each $t$ and $x > x_0(t)$. Here, $x_0(t)$ represents the lower bound of $x$ above which $\lambda_{x,t}$ exhibits a strictly monotonic increase for each $t$. Then, for each $t$, the function $\lambda_t(x)$ defined by

$$\lambda_t : \tilde{X}_t \rightarrow Y, \quad \lambda_t(x) \stackrel{\text{def}}{=} \lambda_{x,t}$$

is an injective (one to one) function of $x$, where $\tilde{X}_t = [x_0(t), \omega]$. Let $\tilde{Y}_t = \lambda_t(\tilde{X}_t)$, then $\lambda_t(x) : \tilde{X}_t \rightarrow \tilde{Y}_t$ has an inverse function $\nu_t(y) : \tilde{Y}_t \rightarrow \tilde{X}_t$ defined on $\tilde{Y}_t$ for each $t$.

Let us define $Y_0$ as follows:

$$Y_0 \stackrel{\text{def}}{=} [y_0, y_1] \quad \text{where} \quad y_0 = \sup_{t \in T_0} \min_{i \in \tilde{Y}_0} \tilde{Y}_t, \quad y_1 = \inf_{t \in T_0} \max_{i \in \tilde{Y}_0} \tilde{Y}_t,$$
Then, we can define $v_{y,t} : Y_0 \times T_0 \rightarrow X_0$ by

$$v_{y,t} \overset{\text{def}}{=} v_t(y)$$

$v_{y,t}$ gives the age $x$ at which the value of the log hazard function is equivalent to a value $y$ at time $t$.

Moreover, we define the following two differential functions by time $t$: (1) $\rho_{x,t}$: the mortality improvement rate and (2) $\tau_{y,t}$: the force of age increase.

$$\rho_{x,t} \overset{\text{def}}{=} -\frac{\partial \lambda_{x,t}}{\partial t} = -\frac{\partial \log \mu_{x,t}}{\partial t}$$

$$\tau_{y,t} \overset{\text{def}}{=} \frac{\partial v_{y,t}}{\partial t}$$

Figure 3 shows a stylized example of the log mortality surface and the above two functions. The blue lines show the log mortality surface in a usual representation, that is, the height from the X-T plane which is determined by $\lambda_{x,t}$. The black point on the log mortality surface is $(x, t, y) = (1, 2, -1.5)$, which could be recognized that the height from the X-T plane is -1.5 when $(x, t) = (1, 2)$. If we travel on the surface with $x$ fixed, the height from the X-T plane will decrease to around -1.86 when $t = 3$, which is shown in a brown arrow. The difference between the two heights corresponds to $-\rho_{x,t}$.

On the other hand, the log mortality surface is also represented by the height from the Y-T plane, which is determined by $v_{y,t}$. In this viewpoint, the black point is recognized that the height from the Y-T plane is 1 when $(y, t) = (-1.5, 2)$. The orange lines on the surface show the contour with $y$ fixed, so we go along these lines when we travel on the surface with $y$ fixed. If we start from the black point again but keep $y$ fixed this time, the height from the Y-T plane will be 3 when $t = 3$, which is shown in a purple arrow in the figure. The difference between the two heights corresponds to $\tau_{x,t}$.

1.3 The Lee-Carter Model (LC)

The Lee-Carter model (abbreviated as LC) is expressed by the following formula (Lee and Carter 1992).

$$\lambda_{x,t} = \log \mu_{x,t} = a_x + k_t b_x$$

where $a_x$ is a standard age pattern of mortality\(^1\)

\(^1\) For the standard pattern, we used the average mortality rates for the whole period when we apply to the actual data in Section 2, and those for the latest 5 years when we project mortality in Section 3. We used the same method for the standard pattern for the LD model.
Taking a partial derivative by time $t$, we obtain the following relationship.

$$\rho_{x,t} = -\frac{dk_t}{dt}b_x = -k'_tb_x$$

This equation shows that the age distribution of $\rho_{x,t}$ is constant in the LC model. If we further assume that $k_t$ is linear over time, $\rho_{x,t}$ is constant over time. Therefore, the LC model works well when the age-specific rate of mortality improvement is considered to be constant over time, that is, the mortality improvement is considered as decline.

1.4 The Linear Difference Model (LD)

Next, we define the Linear Difference model (abbreviated as LD). First, we show the following property of $\tau_{y,t}$ for the two parameter logistic model. It is a theoretical foundation of the LD model.

**Prop 1.** For the two parameter logistic model

$$y = \lambda_{x,t} = \log \frac{\alpha_t \exp(\beta_t x)}{1 + \alpha_t \exp(\beta_t x)} = \log \alpha_t + \beta_t x - \log(1 + \alpha_t \exp(\beta_t x))$$

$\tau_{y,t}$ is a linear function of $x$ for each $t$, i.e.

$$\tau_{y,t} = f_t' + g_t'x$$
Proof.

\[ e^y = \frac{a_t \exp(\beta_tx)}{1 + a_t \exp(\beta_tx)} \]

\[ \iff a_t \exp(\beta_tx) = \frac{e^y}{1 - e^y} \]

With differentiating both sides by \( t \) with \( y \) fixed, we have

\[ a'_t \exp(\beta_tx) + a_t \left( \beta'_t x + \beta_t \frac{\partial x}{\partial t} \right) = 0 \]

\[ \iff \frac{\partial x}{\partial t} = -\frac{a'_t}{a_t \beta_t} - \frac{\beta'_t}{\beta_t} x \]

Since the actual old mortality rates in the HMD and the JMD are estimated using the two parameter logistic model, \( \tau_{y,t} \) for the old age that we use is expected to satisfy this relationship.

Let us define the LD model satisfying this property, i.e. \( \tau_{y,t} \) is a linear function of \( x \) for each \( t \). Then, we can describe the condition for the model in the continuous form as following:

\[ \tau_{y,t} = f'_t + g'_tx \]

This is the differential form. By integrating both sides with \( t \), we obtain

\[ v_{y,t} = f_t + g_t x + a_y \]

where \( a_y \) denotes a standard pattern of inverse log hazard rates.

The Figure 4 shows the stylized example of the LD model. The coloured horizontal arrows in the upper half of the Figure 4 show the amount of shifts of the mortality curve indicated with the black line, which correspond to the \( \tau_{y,t} \). The vertical arrows at the bottom have same lengths as in the upper side with the same colour whose directions are rotated 90 degrees counter-clockwise. The LD model requires that the amount of shifts is a linear function of age, which means the end point of the arrows form a line. The parameter \( g'_t \) means the slope of the above line, so the \( g_t \) means the slope of age increases between time \( t \) and \( t_0 \) (base point of time). The parameter \( f'_t \) is the intercept of the line that cannot be interpreted so easily as \( g'_t \). Therefore, we consider another variable \( S_t \) as a location of the mortality curve instead of \( f'_t \) or \( f_t \). \( S_t \) is defined as the age that the mortality rate equals to 0.5 at time \( t \). We can always covert from \( S_t \) to \( f_t \) using the value of \( g_t \). The Figure 5 shows the stylized example of the effect of change in \( S_t \) and \( g_t \). Assuming that the mortality curve at a base point of year is shown as the black line, the increase of \( S_t \) with \( g_t \) fixed changes the curve into one shown as the red line. Therefore, we can recognize the mortality improvement by the increase of the \( S_t \) as the
shifting of the mortality curve. On the other hand, the decline of $g_t$ with $S_t$ fixed changes the curve into one shown as the blue line, which exhibits some compression features of mortality during the improvement.

In the following section, we fit the LD model to the mortality rates for the European countries. From the estimated parameters $S_t$ and $g_t$, we calculated the slopes of the linear regression trend lines for them to divide the EU countries into three groups.

Figure 4  Stylized example of the LD model
Figure 5  Stylized example of the LD model

Stylized Example of the Effect of change in St and gt

Increase of St
Decline of gt
2 Fitting the LD Model to the Mortality Rates for the EU countries

2.1 Grouping the EU countries

First, we begin with the results of the fitting applying the LD model for Japan. Figure 6 show the trends of $e_0$ and Figure 7 show the trends of $S_t$ and $g_t$ for Japan. Through this period, Japanese life expectancy has prolonged steadily. We can observe that the $S_t$ has also increased steadily in this period, that reveals the shifting feature of the old mortality. On the other hand, $g_t$ has remained almost stationary from 1980 to 2000 that means the shifting is strongly close to parallel shift in this period, although it has decreased before 1980 and after 2000.

Figure 6  Trends of $e_0$ (Female Japan)  
Figure 7  Trends of $S_t$ and $g_t$(Female Japan)

Figure 8 shows the actual log mortality rates for Japan, whereas Figure 9 shows the relative mortality rates compared to the average rates during this period. It is often useful to observe the relative mortality rates to capture the shifting feature, since the age that the amount of improvement is large would move when the mortality curve shifts. From Figure 9, we can observe that the age pattern of mortality improvement shifts to older ages that corresponds to the trends of $S_t$.

Figure 10 and 11 show the relative mortality rates for the LC and LD models. In Figure 10, we see no shifting of mortality improvement since the LC model is essentially decline-type model. The LD model exhibits shifting feature as shown in Figure 11 and the graphs seem to be closer to the actual rates the those by the LC model. This is one of the reason why the LD
Thus, we estimate the slopes of the linear regression trend lines of the model is useful for fitting Japanese mortality rates.

Figure 10 Relative level of Mortality (LC, Female, Japan)

Figure 11 Relative level of Mortality (LD, Female, Japan)

From the observation of the trends of $S_t$ and $g_t$ for Japan, it would be considered important to see how $S_t$ and $g_t$ change over time to measure shifting structures of the mortality curves. Therefore, we estimate the slopes of the linear regression trend lines of the $S_t$ and $g_t$ for EU

...
Figure 12 shows the relationship between the slopes of $S_t$ and $g_t$. First, we can observe that the slope of $S_t$ for Japan is 0.17 and is higher than all the EU countries, which shows the strong shifting feature for the Japanese old age mortality. Also, we can see that the level of $g_t$ for Japan is -0.0034 that is intermediate compared to those for the EU. Through the comparison with Japanese slope parameters and the observation of the two dimensional distribution of the slopes, we divided the EU countries into three groups as shown in Figure 12.

Group 1 consists of the countries whose slopes of $S_t$ are around 0.07 that are positive but lower than Japan, and whose slopes of $g_t$ are around -0.005 that are slightly lower than Japan. Group 1 is the biggest group and would be considered to have the most typical shifting pattern in EU countries. Group 2 consists of the countries whose slopes of $g_t$ are similar or
greater than that of Japan and whose slopes of $S_t$ are positive. Group 3 consists of the other countries.

We will discuss the various features of shifting mortality by the groups with the observation of the parameters of the LD model in the following subsections.
2.2 Group 1

The countries that belong to the Group 1 are Austria, Belgium, West Germany, France, Ireland, Italy, Luxembourg and Portugal.

Figure 13 show the trends of $e_0$ (Female, Group 1) and Figure 14 show the trends of $S_t$ and $g_t$ (Female, Group 1)

Figure 13 and 14 show the trends of $e_0$ and $S_t$ and $g_t$ for the Group 1 countries. Group 1 is characterized as the slopes of $S_t$ are around 0.07 that are positive but lower than Japan and the slopes of $g_t$ are around -0.005 that are slightly lower than Japan. We can observe in Figure 14 that $S_t$ exhibit increasing trends but the amount of increase is smaller than that of Japan. Moreover, $g_t$ show clear decreasing trends that are apparently different from Japan.

Figure 15 and 16 show the the actual and the relative mortality rates for France that belong to Group 1. For France, we can observe the shifting of mortality curve, even though it is not so radically as Japan.

Therefore, the difference of the relative mortality rates between the two models is comparatively small as shown in Figure 17 and 18.
Relative Log Mortality: $\lambda_{x,t} - \lambda_{x,t}^0$

Figure 15 Mortality Rates (Actual, Female, France)

Figure 16 Relative level of Mortality (Actual, Female, France)

Figure 17 Relative level of Mortality (LC, Female, France)

Figure 18 Relative level of Mortality (LD, Female, France)
2.3 Group2

Group 2 includes Czech Republic, United Kingdom, Poland, Denmark, Hungary and Slovakia. They are countries whose slopes of $g_t$ are similar or greater than that of Japan and whose slopes of $S_t$ are positive.

Figure 19 Trends of $e_0$ (Female, Group2) 

![Trends of $e_0$ (Group 2)](image)

Figure 20 Trends of $S_t$ and $g_t$ (Female, Group2)

![Trends of $S_t$ and $g_t$ (Group 2)](image)

From the observation of Figure 20, we observe that $g_t$ remain almost stationary in the first half of the period. This would be one of the reasons why the level of slope of $g_t$ in this group show similar or greater level as that of Japan that remains almost stationary from 1980 to 2000. However, $g_t$s start declining in the second half, which is sillier trends in those in the Group 1. Therefore, it would be better to use trend in the second half for projection purpose for Group 2.

Figure 21, 22, 23 and 24 show the actual mortality rates and the relative rates for actual, the LC and the LD models for Czech Republic. We can observe that the mortality improvement is moderate in the first half of the period, whereas it is larger in the second half. In the second half of the period, $S_t$ increased and $g_t$ decreased, which are similar trends as those in Group 1. Therefore, mortality improvement for Czech Republic could be regarded as stagnation in the first half followed by the improvement in the second half similar to Group 1.
Figure 21  Mortality Rates (Actual, Female, Czech Republic)

Figure 22  Relative level of Mortality (Actual, Female, Czech Republic)

Figure 23  Relative level of Mortality (LC, Female, Czech Republic)

Figure 24  Relative level of Mortality (LD, Female, Czech Republic)
2.4 Group 3

The last group, Group 3, consists of the other countries including Bulgaria, Estonia, Lithuania, Latvia, Netherlands, Spain, Finland and Sweden. This group could be classified into two subgroups. The first one consists of Finland, Spain, Netherlands and Sweden whose life expectancy are relatively higher. The rest countries form second subgroups whose life expectancy are relatively lower. One of the characteristic of this subgroup would be deterioration in the old age mortality.

Figure 25  Trends of $e_0$ (Female, Group5)  

Figure 26  Trends of $S_i$ and $g_t$(Female, Group5))

Figure 27 shows the actual mortality rates for Bulgaria. We can observe that the mortality rates over age 90 deteriorates over time. This fact complicates the pattern of mortality change and adds more difficulty to the mortality modeling, which is exhibited in the Figure 29 and 30.
Figure 27  Mortality Rates (Actual, Female, Bulgaria)

Figure 28  Relative level of Mortality (Actual, Female, Bulgaria)

Figure 29  Relative level of Mortality (LC, Female, Bulgaria)

Figure 30  Relative level of Mortality (LD, Female, Bulgaria)
3 Application to Mortality Projections

3.1 Building an Entire Age Model Using Tangent Vector Fields

In Section 2, we proposed the LD model for the adult mortality. However, we need an entire age model for mortality projection. Therefore, we propose a method to obtain full age pattern by the application of tangent vector fields on the log mortality surface.

We begin with a stylized example of the change in mortality curves shown in Figure 31. Now, we are going to use the LD model for the adult mortality, whose direction of the mortality improvements expressed by the age-increases shown in the red arrows. On the other hand, the mortality improvements in the juvenile mortality are well-modeled by the decline-type models, such as the LC model whose mortality improvements shown in the blue arrows. Here, the arrows express the directions for which the points on the log mortality curves are heading. Mathematically, these arrows are formulated using tangent vector fields on the log mortality surface.

![Figure 31 Change in the Mortality Curves](image)

Here, the following vectors using them

$$\rho(x_0, t_0, y_0) = (0, 1, -\rho_{x_0,t_0})$$
\[ \tau(x_0, t_0, y_0) = (\tau_{y_0, t_0}, 1, 0) \]

are tangent vectors on \( S \) as shown in Figure 32. Each tangent vector defines a tangent vector field on \( S \). If \( \rho_{x,t} \) and/or \( \tau_{y,t} \) satisfy the conditions for the LC and/or the LD model respectively, we can say the upper vector field is corresponding to the LC model, and the lower is to the LD model.

Figure 32 Tangent Vectors on the Log Mortality Surface

Then, from these two vector fields, we can construct one vector field as follows:

- For each point on the adult mortality area, we pick the vector from the LD model.
- For each point on the juvenile mortality area, we pick the vector from the LC model.
- For each point between the two areas, we take a weighted average of the two vectors.

More precisely, we could use a weight function \( w(x, t) \) which takes 0 on young age and 1 on old age, and define a new tangent vector field \( \xi \):

\[ \xi = (1 - w(x, t)) \rho(x, t, y) + w(x, t) \tau(x, t, y) \]

This vector field induces a blended mortality model that has the LC property in age and the LD property in older age. We call it the TVF (Tangent Vector Fields) model. An example of the construction of the TVF model is shown in Figure 33.
3.2 Experimental Mortality Projections for the EU countries

In this subsection, we perform some experimental mortality projections for the EU countries using the TVF model and compare with those by the LC model.

Projection of the mortality is performed by the extrapolation of the parameters $k_t$, $S_t$ and $g_t$. In this study, we simplified the method for projection since our object is evaluating the performance of the LD (TVF) model. We only used linear extrapolation for the projection of the parameters. When the trends of the parameters is apart from linear, we restrict the period for the base data to recent ones. Sometimes this could not be appropriate for forecasts of mortality trend. Therefore, the following results should be recognized as experimental ones.

We performed the simple projections by the groups for the EU countries in this study. As we observed in Section 2, the parameters for the LD model in Group 1 showed linearly trends that the linear projection method is easily applicable. Figure 34, 35 and 36 are the results of the simple projection for West Germany. We can observe that the projected trajectories for both two models are similar, whereas the pattern of $m_x$ are different. In particular, we observe an implausible pattern of mortality by the LC model in 2060 that decreases for some ages. We can see more natural patterns projected by the TVF model. Therefore, the projection by
the TVF model are considered to be more appropriate for the countries in Group 1 from this observation.

Some more consideration are needed for other groups compared to the Group 1. For Group 2, we observed the change of trend between the first and the second half of the period. Therefore, we used the trends in the parameters from 1990 as the base of the projection for this group, except Lithuania for which we used the data from 1995. We applied this simple projection to Group 3. For Group 3, some countries exhibited little difference between the LC and the LD model, and others did less stable results by the TVF method than those by the LC method.

Figure 37, 38 and 39 are the results for Spain from Group 3. For Spain, the results for the two models are not so different.

4 Concluding Remarks

In this study, we applied the LD model to mortality rates for the EU countries and performed experimental mortality projections by the TVF mode that are blended model of the LC and the LD models.

Based on the comparison of the parameters for the LD model with those in Japan, we made three groups of the EU countries, and interpreted the parameters by the groups. We fit the LD and the LC models and compared the performances of the two models. We also performed simple experimental mortality projections based on the observation of the trends of the parameters. We found that the projection by the TVF model worked well for the countries
in Group 1, whereas some more consideration are needed for other groups. We also found that the LC model produces an implausible pattern of mortality which decreases for some ages, whereas the TVF model produces more natural patterns. This could be considered as an advantage for applying the TVF model for mortality projections to the EU countries.

There are some caveats for the projections in this study. The projections performed in this study should be understood as a preliminary work, and should not be recognized as some practical forecasts of the future mortality. There should be more considerations about the methods for extrapolations, the connectivity of the jump-off year, and so on. Further researches are needed for projecting mortality for all the EU countries.

References


Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de.

