Event-Centered Demographic Methods: Theory and Examples

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Introduction
Demographic events, including births and deaths, are by definition at the center of demographic measures, so much so that demographic rates can be regarded just as a way of weighting events according to the population exposed to risk. Demographic methods can then be viewed as a way to handle data on events to produce meaningful summary measures of interest.

The key element here is what we understand as meaningful measures. Traditional theoretical statistics and econometrics derive meaning by working from a hypothetical model generating the data. Traditional demographic methods derived meaning by summarizing patterns for complete populations based on universal coverage of censuses and vital statistics. The meaning is derived because the population of interest is being described. In that respect, demographic methods seem to be model free, but they are generally not. Models are needed in the background to separate concurrent phenomena such as fertility, mortality and migration (Henry, 1959). The standard life table approach, the central element for demographic methods, depends on meeting assumptions such as homogeneity, independence or continuity to separate concurrent phenomena (Wunsch, 2006), and demographers have long been aware that when those assumptions are not met composite population patterns can be very different from individual risk patterns (Vaupel and Yashin, 1985).

Instead of reconciling population and individual patterns, event-centered demographic methods are introduced that take as given a set of events of interest (births, deaths, marriages, ...) observed in a population. The complex process of concurrent phenomena that generates the events is left unspecified. The focus is on measuring the importance of the events for a population of interest. Putting in relation events (occurrences) and exposure is a way of deriving rates to estimate a model, but it is also just a sensible way of
understanding how frequent the events are. But using the population at risk as the measuring stick is not the only sensible way to go. General event-centered concepts are introduced by which events are measured up against an arbitrary reduction factor. In doing so, they are a generalization of what demographers have called rates of the second-kind (Calot, 2001b), or, in the French tradition, reduced events (événements reduits) (Henry, 1963, 1972; Wunsch, 1968, 2001; Sardon, 1993). There is also a connection with inverse probability weighting (Wooldridge, 2007), and synergies with other model free and data intensive approaches such as the bootstrap (Efron and Tibshirani, 1994).

There is much to be gained in having a general formulation that covers all possible cases of event-centered methods, since the choice of reducing factors is left up to the researcher. In this concept paper I focus on a particular application based on using cohort size at birth as the reducing factor for any individual experiencing a demographic event. Different applications of the concepts include the analysis of reduced-deaths and death cohorts (Ortega, 2012), reduced-births and birth replacement sum, an event-centered generalization of the birth replacement ratios (Ortega, 2006; Del Rey and Ortega, 2011), and the analysis of reduced-presence at population and census counts. What makes the choice of birth cohort size as a reducing factor interesting is that it provides dual measures to closed population measures. In a closed population the only entries to the population happen at birth, contrary to a real open population subject to migration. The same computations that have a meaning in a closed population, like, for instance, lifetable measures, have a dual meaning that incorporates the effect of migration when applied to an open population.

Besides the use of a particular reducing factor, I also highlight how the new methods encompass standard demographic rates and event-history concepts like the Nelson-Aalen and Kaplan-Meier estimators as particular cases of reduced events.

**General Event-Centered Concepts**

The event-centered perspective reflects that indicators are defined from event-level concepts, in contrast with the traditional demographic perspective that we could call person-centered. The person-centered perspective on fertility, for instance, focuses on how many children a woman might bear, whereas the event-centered perspective is based on the idea that each event contributes to the phenomenon of interest. The contribution of a single event is measured with respect to the relevant terms of comparison, for instance, the number of women in the age-group: the reduction factor. The choice of a reduction factor determines the particular interpretation of the reduced
event defined from it. The person-centered perspective is forward-looking, in the sense that it sees the person at risk of experiencing the event. The event-centered perspective focuses on the end result, the event, irrespective of whether each event was more or less likely a priori. Many standard demographic measures can be interpreted from both perspectives; others only have meaning from one of the perspectives.

A general formulation that does not make explicit what the events are is adaptable to different processes generating the events. The methods are equally valid for vital events from vital registration, retrospective survey data, Census or population counts, and follow-up methods. They are also equally valid for different populations of reference and types of events. The general definition of reduction factors with the only restriction that they are a positive makes it possible to define very different reduced events from the same events, such as replacement densities, net maternity function, cohort net maternity or fertility rates.

The main logic of demographic methods is the analysis of the dependence of aggregate population measures, such as the number of births or the mean length of life, on some basic dimensions, such as age, duration since a given event, or period. The most widely used measures refer to the Lexis space defined by the combination of a period, \( t \in \mathbb{R} \), and age, \( x \in [0, \omega] \), where \( \omega \) is an upper bound on the length of life. A lexis point, \( a \), is defined by a valid \((x, t)\) combinations in the Lexis space, \( \mathcal{L} \).

**Definition 1: Reduced-event:** A reduced-event, \( \varepsilon \), is defined by the pair \((a, \rho) \in \mathcal{L} \times \mathbb{R}^+ \), where \( \rho \) is the reduction factor, strictly positive.

**Examples:** A reduced birth in the standard Lexis space is defined by a combination of the time of birth and age of the mother at birth, and a reduction factor that might be the number of women in the birth cohort corresponding to the year of birth of the mother, \( B(\text{floor}(t - x)) \), or the number of women reaching age \( a \) in the population within the time interval \([t - 1/2; t + 1/2]\). Reduced presence in a Census is defined by the Lexis point age-at-census and birth cohort size in the year of birth.

**Definition 2: Lexis-Regions:** A lexis-region, \( A \), is a subset of the Lexis space of non-null Lebesgue measure, \( \mu(A) > 0 \).

**Examples:** Examples of Lexis-regions include Lexis rectangles, \([x, x + 1) \times [t, t + 1)\), of measure 1, Lexis triangles, of measure \(1/2\); Lexis period stripes, \([0, \omega] \times [t, t + 1]\), and cohort stripes, \(\{(x, t): t - x \in [c, c + 1]\}\), both of measure \(\omega\). See Calot (2001a) and

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1 The Lexis space so defined is only one among the possibilities for general reduced events. In general, it might be composed of any dimensions of interest, including duration, geographic location, etc. The point and area notation are valid for any such general Lexis space.
Caselli and Vallin (2006) for illustrations. A Lexis-region of dimension one can also be defined for population counts, referring to population counts at a given moment in time, \( t \), or the moment in which a particular age-line, \( x \), is reached as a subset of a Lexis line.

**Definition 3: Sum of reduced-events:** Given a finite set of reduced events \( \varepsilon = \{\varepsilon_1, ..., \varepsilon_n\} \) defined in \( \mathcal{L} \times \mathbb{R}^+ \), the sum of reduced events for a Lexis region \( A \) is given by

\[
S(A, \varepsilon) = \sum_{a_i \in A} \frac{1}{\rho_i} = \sum_i \frac{I_A(a_i)}{\rho_i}
\]

where \( I_A(a_i) \) is an indicator function equal to 1 if \( a_i \in A \), and 0 otherwise.

**Examples:**

- For reduced births defined by birth cohort size, the sum of reduced births over period or cohort stripes produces a measure of period or cohort replacement that includes mortality and migration, called the *Birth Replacement Sum*. Cohort sums are particularly simple, since all events share the same reduction factor, birth cohort size. The cohort birth replacement sum has therefore a particularly simple form: the sum of the births born to women born in the same year at different ages divided by the number of women in the birth cohort. It was described as a potential method by Shyrock and Siegel (1973: 537), but to my knowledge it had not been applied anywhere.
- For reduced births defined by the number of women reaching age \( a \), the sum of reduced births corresponds to a period or cohort total fertility rate (TFR), depending on whether \( A \) is a period or cohort stripe. When the sums are defined over a truncated stripe, one gets truncated sums, that are interesting to assess fertility trends from survey data (Brass and Juárez, 1983; Moultrie et al., 2012). Figure 1 compares the cohort Birth Replacement Sum and the cohort TFR for a sample of countries.

\[\text{footnote:} \text{Note that, while conceptually equivalent to a TFR, the reduced event has been defined by a criteria of local exposure, instead of exposure in a fixed interval as traditionally defined. That is just one option. See the section on aggregate measures for a standard computation of TFR.}\]
In retrospective sample surveys with birth histories, all events refer to a closed cohort of women followed over time. Exposure, therefore, is constant, and the two approaches defined above lead to the same result: the Total Fertility Rate.

**Definition 4: Reduced-event densities.** Given a finite set of reduced events defined in $\mathcal{L} \times \mathbb{R}^+$, the density of reduced events for a Lexis region $A$ is given by $\Delta(A, \varepsilon) = \frac{S(A, \varepsilon)}{\mu(A)}$.

**Examples:** For reduced deaths defined by birth cohort size, the density of reduced deaths defined for an age-cohort Lexis parallelogram has life table deaths, $d_x$, as a closed-population dual: For a closed population, cohort reduced death densities coincide exactly with cohort life table deaths. To the extent that there is net migration, densities will be bigger (or smaller) than life table deaths. Reduced death densities observed along period lines are affected by net migration and, specially, by mortality trends in a mortality equivalent of the cohort translation problem: If mortality is going down over time, the sum of period reduced deaths can be far below one since the people that would be expected to die at higher ages are not present in the population, having died earlier on. Figure 2 compares period reduced death densities with period and cohort life table deaths. Net migration explains the difference between reduced deaths and cohort life table deaths. Mortality trends are responsible for the large gap between cohort and period life table deaths. Note that reduced deaths, in contrast to cohort life table deaths, do not require observing the cohort since birth.

Densities of birth-cohort reduced births provide an age-specific measure of population replacement. If reduced births are defined by local exposure, the reduced birth density is an estimate of the age-specific fertility rate.

**Theorem 1: Sums of reduced-events from Reduced-event densities.** Given a finite partition of a Lexis region $A$, $P(A) = \{A_1, \ldots, A_m\}$ of a Lexis region $A$, $\sum_{i=1}^{m} \mu(A_i) \cdot \Delta(A_i, \varepsilon)$

Interpretation: The result shows that it is possible to derive sums of reduced-events from densities. This is the usual demographic practice of deriving quantum indicators from sums of age-specific rates. Note, however, that this is just an option. It is possible to derive sums from rates, but it is not necessary. Sums can be obtained directly by adding up all the reduced events in the Lexis region of interest. The next section develops the case of estimation based on aggregate data.

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3 The properties defining the partition are $\bigcup_{j=1}^{m} A_j = A$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$
Note: The finite partition of the Lexis region is arbitrary and does not need to coincide with the areas, if any, used to derive the reduced-event. Event reduction and aggregation are two different operations that jointly define event-centered measures. Theorem 1 guarantees that no matter the partition, the sums are consistent. For this reason the sums are not a function of the partition.

Definition 5: Mean age at reduced event. For Lexis spaces having an age dimension, $x$, the mean age at a reduced event for a Lexis region $A$ is given by $T(A, \varepsilon) = \frac{\sum_{a_i \in A} x_i \rho_i}{S(T, \varepsilon)}$. The mean age at reduced event is an indicator of tempo. As was the case for $S(A, \varepsilon)$, $T(A, \varepsilon)$ is obtained directly from the reduced events. There is no need to derive densities first, and doing so implies a loss of information regarding the age location of the events.

Example: The cohort mean age at reduced death using the birth cohort as the reduction factor is an open-population dual of life expectancy, combining length of life and presence. A simple adjustment (Ortega, 2012) provides an indicator of cohort length of life that does not require the use of any lifetable.

Definition 6: Standardized reduced events. It is often useful, in particular for comparison of tempo change over time or over cohorts, to have reduced events that add-up to one in a particular Lexis region. For a given reduced birth, $\varepsilon_0 = (a_0, \rho_0)$, and a standardizing region $A$, such that $a_0 \in A$, the standardized reduced birth is given by $\theta(\varepsilon_0, A) = \left( a_0, \frac{\rho_0}{S(A, \varepsilon)} \right)$. The standardized reduced birth is a reduced birth itself with the property that the sum of reduced births is one in a set of interest, such as a Lexis cohort stripe.

Example: Reduced deaths defined by birth cohort size, standardized across cohort lines, produce reduced deaths defined by death cohort size, where death cohort size is the number of people dying in a territory born in the same year (Ortega, 2012). Such reduced deaths can be used to explore changes in mortality and migration over time, or to define new sums and densities over period lines that are less affected by migration.

Applications: Since standardized reduced events add up to one, they can be interpreted as a probability distribution, itself a realization of an underlying process that is left unspecified. Such a process is the combined result of the process generating the events and the reduction factors used as weights. Marginal distributions, quantiles and mean ages can be of interest in many contexts, particularly for non-universal non-renewable events (e.g: mean and median ages at first birth or first marriage).
Aggregate event-centered measures

While event-centered measures are defined at the most general event-level, standard demographic measures are traditionally computed from aggregate data. In aggregate data the exact location of events is not known, only event counts defined over a partition of the Lexis space (i.e: Lexis triangles). This restricts the possible measures that can be defined in an important way: If aggregate data is only available for a given finite partition \( P(A) \) of the Lexis region of interest \( A \), this implies two restrictions:

- **Reduction**: Since all events belonging to the same partition share the same information by definition, all are treated alike. This means that the same reduction factor, \( \rho_j \), is used for all the events in \( A_j \), \( \rho_j = \rho(A_j) \), so that \( \rho = \{ \rho_1, \ldots, \rho_m \} \) is a vector including all the reduction factors. They will generally be a function of the Lexis location. See how this restricts considerably the possibilities of defining reduced events that can only be defined at the subregion level.

- **Aggregation**: Sums of reduced events for a region of interest can be obtained based on theorem 1 from the reduced-event densities defined over the partition \( P(A) \). Since the reduced-event is the same for all events, definition 3 applied to this case leads to \( S(A, \varepsilon) = \frac{\sum l_{A_j}(a_i)}{\rho_j} = \frac{n_j}{\rho_j} \) where \( n_j = \sum l_{A_j}(a_i) \). Reduced-event density is given by \( \Delta_j = \frac{n_j}{\mu(A_j)\rho_j} \).

**Theorem 2: Sum of reduced-events from aggregate information for a partition \( P(A) \)**

Given a finite partition of a Lexis-region of interest \( A \) for which aggregate events are available, an event count \( n_j = \sum l_{A_j}(a_i) \) given by \( N = \{ n_1, \ldots, n_m \} \), and reduction factors equal to \( \rho = \{ \rho_1, \ldots, \rho_m \} \), the sum of reduced events is given by \( S(A, \varepsilon) = \sum_{j=1}^{m} \frac{n_j}{\rho_j} \).

**Example**: Standard TFR as calculated from aggregate data is an example. The partition is defined by the data (Lexis-triangles, rectangles, ...). The count of the number of births observed in each of the sub-regions of the partition, \( N \), is the only information regarding the events. The reduction factors in each lexis sub-region are the same and given by \( \frac{E_j}{\mu(A_j)} \), where \( E_j \) is a measure of exposure or women-years lived in the Lexis sub-region.

**Interpretation**: This definition is adaptable to all standard demographic measures defined as a sum from aggregate data. Note also, that the restriction to use the same reduction factor within the subregion is less stringent than it appears. It does not mean necessarily
that if the information were available at the event-level, the same reduction factors would be used. For instance, in a follow-up study of a cohort the reduction factors that would be assigned might follow a sequence 10, 9, ... However, since we lack information on the exact order of each event, we can define an average reduction factor that would be the same for all events. That reduction factor can be obtained a posteriori from the reduced-event density.

**Closed population and stable population duals**

It has been shown in previous examples that it is possible to define different reduced events from the same set of events by using different reduction factors. An example is the use of local or region-specific measures of exposure, that lead to alternative definitions of standard demographic rates, or the use of birth cohort size, that leads to open-population based densities that incorporate the effect of net migration. The basic idea is that in a closed population model births constitute the sole means of entry to a population. The number of births in a certain period are given by $B(t)$. The only exits are given by deaths, $D(t)$. Population growth would be given by the difference $B(t) - D(t)$. In contrast, in an open-population, population growth is not equal to $B(t) - D(t)$, it also includes net migration $NM(t)$. This is a first example of a closed-population dual: the same calculation, namely, looking at population change, has one interpretation in a closed population and a different one in an open population that includes migration.

**Definition 7:** Closed population dual reduced event: For every reduced event that we can define, $(a, \rho)$ a closed population dual, $(a, \rho^C)$ is such that the two reduced events are equal in a closed population with no migration, and generally different in the presence of migration. Reduced-event densities and sums obtained from the closed population dual reduced event are also called closed population duals of those obtained from $\epsilon$.

**Example:** Take birth cohort size, $B(t - x)$ as a reducing factor. The open-population dual is a reduction factor such that it is equal to $B(t - x)$ in the case of a closed population, but generally different when there is migration. In order to derive a closed population dual one needs to consider that the only exits in a closed population are given by deaths, $D(x,t)$. The answer is what would have been the observed population in the absence of migration. If cohort mortality from birth to age $x$ is given by $l(x, t)$, then a closed population dual would be $\frac{P(x, t)}{l(x, t)}$. In the case of a closed population, $P(x, t) = B(t - x) \cdot l(x, t)$ ensuring that definition 7 is satisfied.
**Application:** The use of $B(t - x)$ as a reducing factor of births tabulated by mother’s age leads to birth replacement densities and the birth replacement sum. When reduction factors are defined for annual cohort-period lexis regions, a closed population dual is $\frac{E(x, t)}{L(x, t)}$ where $E(x, t)$ is observed exposure, and $L(x, t)$ is the cohort person-years lived. Note the relationship between the closed-population dual and the standard net maternity function, which would be obtained by using $\frac{E(x, t)}{L^P(x, t)}$ as the reduction factor, where $L^P(x, t)$ is the period lifetable person-years lived. In such a way reduced-birth densities are equal to $L^P(x, t) \cdot F(x, t)$. This closed-population dual can then be described as cohort net maternity.

**Non-uniqueness of closed population duals:** Closed population duals are not unique. For instance, a second closed population dual to the use of $B(t - x)$ is the use of $B^{rb}(t - x)$, where $B^{rb}$ is the cohort size at birth in the respective region of birth of the person. In a closed population $B^{rb}(t - x) = B(t - x)$, whereas in an open-population they are different for those individuals who were not born in the region. Each closed population dual provides an assessment of the effect of population movements compared to natural factors. For instance, using $B^{rb}$ and aggregating over all events sharing the region of birth a Birth Replacement Sum is obtained according to the region of birth instead of region of residence.

**Definition 8: Stable population dual reduced event.** For every reduced event that we can define, $(a, \rho)$ a stable population dual, $(a, \rho^S)$ is such that the two reduced events are equal in a closed stable population with no migration, and constant fertility and mortality rates.

**Theorem 3:** Every closed population dual is a stable population dual (the reverse is not generally true).

**Example:** In the application of the closed population dual it has been shown that $\frac{E(x, t)}{L(x, t)}$ provides a closed population dual, with $L(x, t)$ referring to cohort birth replacement. In a stable population, cohort mortality is equal to period mortality so that a stable population dual is given by $\rho^S(x, t) = \frac{E(x, t)}{L^P(x, t)}$. This stable population dual leads to period net maternity as a reduced-birth density dual to birth replacement, and the Net Replacement Ratio, NRR, as a stable population dual to the Birth Replacement Sum.
The following table gives the three most basic reduced-events that use $B(t - x)$ as the reduction factor, and the interpretation of their densities, sums and mean-ages, as well as their stable-population dual:

<table>
<thead>
<tr>
<th>EVENT (Numerator)</th>
<th>Reduced-event Density (and dual)</th>
<th>Sum of reduced events (and dual)</th>
<th>Mean age at reduced-event (and dual)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population by age</td>
<td>Reduced presence $RP(x,t)$</td>
<td>Aggregate presence</td>
<td>Mean-age at presence</td>
</tr>
<tr>
<td>$P(x,t)$</td>
<td>(lifetable person years lived, $L_x$)</td>
<td>(Life expectancy at birth, $e_0$)</td>
<td>Mean-age of stationary population</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Births by age</td>
<td>Reduced births $RB(x,t)$</td>
<td>Birth Replacement Sum</td>
<td>Mean age at reduced-birth</td>
</tr>
<tr>
<td>$B(x,t)$</td>
<td>(Age-specific maternity, $L_x F_x$)</td>
<td>(Net reproduction rate, $R_0$)</td>
<td>(Mean age of maternity function)</td>
</tr>
<tr>
<td>Deaths by age</td>
<td>Reduced deaths $RD(x,t)$</td>
<td>Death cohort</td>
<td>Mean age at reduced-death</td>
</tr>
<tr>
<td>$D(x,t)$</td>
<td>(lifetable deaths, $d_x$)</td>
<td>(Always equal to 1)</td>
<td>(Life expectancy at birth, $e_0$)</td>
</tr>
</tbody>
</table>

The most basic such reduced-event is presence, defined by a population count. In most occasions $t$ is fixed while $x$ varies. This is the case in censuses or population counts. Reduced-presence compares the population present to the population that was born into the cohorts. If usual annual categories are used it would give the population present of age $x$ as a share of those born in $t-x$. In a closed population that fraction is given by the lifetable person years lived, $L_x$, or at the instant level, life table survivors, $l_x$. In such a population reduced-presence declines monotonously with age for a cohort from a level of 1 at birth to 0 at $\omega$. In an open population this is not generally the case. To those present in a previous period as added those immigrating, while those emigrating loose presence. By defining reduced presence with respect to birth cohort size, emigration is treated as a death, while immigration would be something as a rebirth, with the possibility that immigration might be strong enough to push presence over one. This indicates a strong immigration, a net migration that is stronger than mortality. In contrast, in regions of emigration presence can fall quite rapidly to low levels. In figure 3, cohort presence is
shown for a sample of cohorts in four Spanish provinces, Madrid and Barcelona, metropolitan regions receiving large immigration, Guadalajara and Zamora, traditional sending regions turned to receiving, particularly the former.

**Relationship between ratios and sums**

Following an interpretation of the Total Fertility Rate due to Calot (1984) as the ratio between the number of births and a weighted mean of the size of the female population classified by age in the reproductive range, Ortega (2006) defined a Birth Replacement Ratio as the ratio between the number of births and weighted mean of the number of female births in the past using the same weights as in the TFR calculation. Ratios can be seen as a general estimation technique defined by the total number of events, \( n \), and information for a finite partition of the Lexis region \( A \),

**Definition 9: Demographic Ratio**

Given a total number of events, \( n \), a finite partition of the Lexis region \( A \), \( P(A) = \{A_1, \ldots, A_m\} \), a vector of standardized densities, \( \bar{\Delta} = \{\bar{\Delta}_1, \ldots, \bar{\Delta}_m\} \) such that \( \sum_{i=1}^{m} \mu(A_i) \cdot \bar{\Delta}_i = 1 \), and a vector \( K = \{K_1, \ldots, K_m\} \) of group size, a ratio indicator, \( R \), is defined as

\[
R(n, P(A), \bar{\Delta}, K) = \frac{n \sum \mu(A_j) \cdot \bar{\Delta}_j \cdot K_j}{\sum \mu(A_j) \cdot \bar{\Delta}_j \cdot K_j}
\]

**Examples:**

Following Calot’s (1984) interpretation of TFR, and calling it can be put as a ratio:

\[
TFR(P(A), B, E) = \sum_{j=1}^{m} \mu(A_j) \frac{B_j}{E_j} = \frac{\sum B_j}{\sum E_j TFR(P(A), B, E) \cdot E_j}
\]

Identifying terms, \( n = \sum B_j \), the sum of births in each of the Lexis sub-regions,

\[
\bar{\Delta}_j = \frac{B_j}{E_j TFR(P(A), B, E)} \quad \text{and} \quad K_j = \frac{E_j}{\mu(A_j)}
\]

It is simple to show that they satisfy the requirements of definition 9 so that TFR can be seen as a ratio of the number of events to a weighted average of exposures per area.

- The Birth Replacement Ratio (Ortega, 2006; Del Rey and Ortega, 2011) is an indicator of birth replacement that uses the same definition given above for TFR as a ratio but using, instead a different group size indicator. If the partitions \( A_j \) all belong to the same
cohort of birth, \(Coh(A_j)\), defined at the point level at \(t - x\), and having a length in years denoted by \(l(A_j)\) (e.g. If a five-year cohort-period parallelogram is used, length would be equal to five), then the BRR is given by the ratio

\[
R \left( \sum B_j, P(A), \{ \frac{B_i}{E_j TFR(P(A),E_j)} \}, B \left( Coh(A_j) \right) \right),
\]

that is, instead of using a weighted average of exposures as for TFR, a weighted average of cohort size at birth is used (standardized in case subregions have different cohort length). Ortega (2006) shows the possibility to decompose the difference between TFR and BRR in a product of a cohort mortality factor and a net migration factor. This decomposition relies on the use of the same \(\Delta_i\) as for TFR.

**Interpretation:** The two examples illustrate two different cases: TFR is previously defined as a sum, but it can be given an interpretation as a ratio. BRR is only defined as a ratio, since it takes as given the TFR weights or standardized densities. This facilitates comparison, but it means that BRR cannot be decomposed additively in terms of the contribution of each Lexis sub-region. This is not the result of chance: it is only possible to decompose additively indicators than can be put in the form of a sum of reduced events such as TFR or the BRS. This requires using the standardized densities of reduced events as the standardized densities. The cost is that the weights are generally different for each indicator.

**Theorem 4:** Ratio form of a sum of reduced events. Given a lexis region, \(A\), and a finite set of reduced events \(\varepsilon\), such that the reduction factors \(\rho_i\), take unique values within each of the subregions, \(A_j\) of a partition of \(A\), \(\rho_j = \rho(A_j)\), so that \(\rho = (\rho_1, ..., \rho_m)\) and \(\Delta = (\Delta(A_1, \varepsilon), ..., \Delta(A_m, \varepsilon))\) then the sum of reduced events can be obtained as the ratio

\[
R \left( \sum I_A(a_j), P(A), \Delta_{S(A,\varepsilon)}, \rho \right)
\]

**Interpretation:** Ratios provide a more general class of estimates that only require information on the total aggregate number of events and the standardized densities. Many demographic indicators belong to this category that is closely connected to indirect standardization.

It is very often the case that one wishes to compare the ratios defined from the same total number of events and partition, but based on a different set of standardized weights. An example is the comparison of the Birth Replacement Sum, that according to theorem 4 is equal to

\[
BRS = R \left( n, P(A), \frac{B_i}{\mu_j B(Coh(A_j))}, B \left( Coh(A_i) \right) \right),
\]

and the BRR as originally
defined by Ortega (2006), $R\left(n, P(A), \frac{B_i}{E_i \cdot TFR(P(A), B, E)}, B(Coh(A_i))\right)$. Fortunately, Vaupel and Zhang (2012) provide a theorem proving the difference between two weighted averages defined by a different vector of weights. Theorem 5 provides the application to ratios.

**Theorem 5: Comparison of ratios.** Given two ratios, $R_1(n, P(A), \tilde{\Delta}_1, K)$ and $R_2(n, P(A), \tilde{\Delta}_2, K)$ that only differ in the vector of standardized densities, it is possible to write $R_2$ as a function of $R_1$ as follows:

$$R_2 = \frac{n}{n R_1} + \frac{cov(\mu \cdot \tilde{\Delta}_1, \frac{\tilde{\Delta}_2}{\tilde{\Delta}_1})}{\tilde{\Delta}_2 \tilde{\Delta}_1}$$

Where $\tilde{\Delta}_2 \tilde{\Delta}_1$ is the mean of the ratios of standardized densities.

**Conclusions**

Event-centered demographic measures provide a generalization of demographic measures that is defined at the event-level, taking events as given. This is in line with demographic applications that attempt to describe what is happening in a real population. It is unlike theoretical statistics, that assumes that a certain model of behavior has generated the data and the purpose is estimating the parameters of the model. It is shown that standard demographic measures defined as sums of rates all fit into an event-centered definition, with event-centered measures being far more general in allowing the separation of two-stages: reduction, concerned with the definition of reduced events for each of the individual events, and aggregation, concerned with the aggregation of events into sums of reduced events, densities of reduced events, mean ages of reduced events, ...

Particularly useful examples of new event-centered measures are those defined by using cohort size at birth as the reducing factor. It is shown that the reduced events so defined differ from traditional occurrence-exposure rates in being affected by net migration. Closed population and stable population duals are defined as measures that would be equal to the measure under consideration in a closed (resp. stable) population but differ in an open population. This provides a useful way to interpret the meaning of the indicators. Birth replacement, reduced deaths and reduced presence are the three most basic such concepts based on births, deaths and population counts respectively.
Finally, event-centered measures have been compared to ratios. Ratios provide an even more general kind of estimates than event-centered sums of reduced events. It is shown that every event-centered sum can be interpreted as a ratio, but that the use of different weighting factors leads to different ratios. The effect of choosing different weights is predictable and depends on the covariance between the reduced events and the ratio of standardized reduced densities.

**Proofs of Theorems**

**Theorem 1.** Sums of reduced-events from Reduced-event densities.

\[
S(A, \varepsilon) = \sum_{i=1}^{m} \mu(A_i) \cdot \Delta_i(A_i, \varepsilon) = \sum_{i=1}^{m} \mu(A_i) \cdot \left( \sum_{j=1}^{m} \frac{I_{A_i}(a_j)}{\rho_j} \right) = \sum_{j} I_A(a_j)
\]

**Theorem 2:** Sum of reduced-events from aggregate information for a partition \(P(A)\).

Since reduction factors are unique within each of the Lexis sub-regions, the sum of reduced events in definition 3 for a subregion \(A_j\), is given by

\[
S(A_j, \varepsilon) = \sum_{i} I_{A_j}(a_i) \cdot \frac{n_j}{\rho_j}
\]

For the complete area \(S(A, \varepsilon) = \sum_{j=1}^{n} \frac{n_j}{\rho_j}\)

**Theorem 3.** Every closed population dual is a stable population dual.

Since by definition 7, in any closed population \((a, \rho) = (a, \rho^c)\), this also holds for a closed stable population that is also closed. It is obvious to observe why the reverse is not true.

**Theorem 4:** Ratio form of a sum of reduced events. First, we observe that the definition 9 of a ratio is fulfilled since \(\sum_{i=1}^{m} \mu(A_i) \cdot \frac{\Delta_i}{S(A, \varepsilon)} = 1\) by application of theorem 1. Second, the sum of reduced events is equal by theorem 2 to \(S(A, \varepsilon) = \sum_{j=1}^{n} \frac{n_j}{\rho_j}\). The application of definition 9 shows that \(S(A, \varepsilon) = R\left(\sum_{i} I_{A_i}(a_i), P(A), \frac{\Delta}{S(A, \varepsilon)}, \rho\right)\).

**Theorem 5:** Comparison of ratios. Vaupel and Zhang (2012, eq. 3) provide a decomposition of the difference of weighted averages that can be applied to the difference in the denominators of \(R_1\) and \(R_2\). By plugging in definition 9, the result is obtained.

**REFERENCES**


Fig 1. Cohort Birth Replacement Sum and Fertility

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<th>Country</th>
<th>TFR</th>
<th>FBRS</th>
<th>Cohort TFR + FBRS</th>
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<td>3.5</td>
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<tr>
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<td>3.5</td>
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<tr>
<td>Japan</td>
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<td>2.0</td>
<td>3.5</td>
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Own elaboration based on HFD and supplementary birth data

Figure 2. Period Reduced Death Densities and Life Tab

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<th>Country</th>
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<th>Period lifetable deaths</th>
<th>Cohort lifetable deaths</th>
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<td>Japan</td>
<td>0-20</td>
<td>0.1</td>
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<td>0.3</td>
</tr>
</tbody>
</table>

Females, 1970. Own elaboration based on HMD and supplementary data.
Source: Spanish data for a sample of provinces and birth cohorts. Own elaboration based on Ortega and Sánchez-Barricarte (2013).